Integracija racionalnih funkcija

Neka su $P_n(x)$ i $Q_m(x)$ polinomi n-tog, odnosno m-tog stepena. Tada je f-ja $R(x) = \frac{P_n(x)}{Q_m(x)}$ racionalna f-ja.

Racionalne f-je se nazivaju pravim racionalnim f-jama ako je $n < m$, u suprotnom se nazivaju nepravilnim. Tada svaku nepravilnu f-ju možemo zapisati kao:

$$R(x) = \frac{P_n(x)}{Q_m(x)} = S_k(x) + \frac{T_\ell(x)}{Q_m(x)}, \qquad (7)$$

gdje je $S_k(x)$ cijeli dio razlomka P_n/Q_m, a $T_\ell(x)$ ostatak tog razlomka ($T_\ell(x)$ je polinom stepena $\ell < m$).

Primjer $R(x) = \frac{5x^3 - 3x^2 + 6x - 7}{x^2 - 4} = 5x - 3 + \frac{26x - 19}{x^2 - 4}$

Saglasno (7) svaki integral nepravilne racionalne f-je se svodi na integral prave racionalne f-je.

Smatraćemo da je $R(x) = \frac{P_n(x)}{Q_m(x)}$ - prava racionalna f-ja.

Među pravim racionalnim f-jama razlikujemo četiri tipa ~~na~~ takozvanih najprostijih racionalnih f-ja (ili najprostijih razlomaka)

I $\quad \frac{A}{x-a}$ $\qquad$ II $\quad \frac{A}{(x-a)^k}$ $\quad k \geq 2$

III $\quad \frac{Mx+N}{x^2+px+q}$ $\qquad$ IV $\quad \frac{Mx+N}{(x^2+px+q)^k}$ $\quad k \geq 2$

gdje su A, M, N, a, p, q konstante, k cio pozitivan broj, diskriminanta $D = p^2 - 4q < 0$.

Naćimo integrale ovih f-ja.

$$\mathrm{I}.\quad \int \frac{A}{x-a}\,dx = A\int \frac{d(x-a)}{x-a} = A\ln|x-a| + c$$

$$\mathrm{II}.\quad \int \frac{A}{(x-a)^k}\,dx = A\int (x-a)^{-k}\cdot d(x-a) = \frac{A}{1-k}\cdot\frac{1}{(x-a)^{k-1}} + C.$$

Za integraciju $\underline{\mathrm{III}}$ i $\underline{\mathrm{IV}}$ preobražujemo polinom (trinom) x^2+px+q u sledeći oblik

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p^2}{4}\right)$$

Uzmimo sada da je $x+\frac{p}{2}=t$, $q-\frac{p^2}{4}=a^2>0$. Dobijamo:

$$\underline{\mathrm{III}}\quad \int \frac{Mx+N}{x^2+px+q}\,dx = \int \frac{M\left(x+\frac{p}{2}\right) + N - \frac{Mp}{2}}{\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}}\,dx =$$

$$= M\int \frac{t\,dt}{t^2+a^2} + \left(N-\frac{Mp}{2}\right)\int \frac{dt}{t^2+a^2} = \frac{M}{2}\ln(t^2+a^2) +$$

$$+ \frac{2N-Mp}{2}\,\frac{1}{a}\operatorname{arctg}\frac{t}{a} + c =$$

$$= \frac{M}{2}\ln\left(x^2+px+q\right) + \frac{2N-Mp}{\sqrt{4q-p^2}}\cdot\operatorname{arctg}\frac{2x+p}{\sqrt{4q-p^2}} + c$$

$$\underline{\mathrm{IV}}.\quad \int \frac{(Mx+N)\,dx}{(x^2+px+q)^k} = \int \frac{M\left(x+\frac{p}{2}\right) + \left(N-\frac{Mp}{2}\right)}{\left[\left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p^2}{2}\right)\right]^k}\,dx =$$

$$= M\int \frac{t\,dt}{(t^2+a^2)^k} + \frac{2N-Mp}{2}\int \frac{dt}{(t^2+a^2)^k}$$

$$\int \frac{t\,dt}{(t^2+a^2)^k} = \frac{1}{2}\int \frac{d(t^2+a^2)}{(t^2+a^2)^k} = \frac{1}{2(1-k)}\cdot\frac{1}{(t^2+a^2)^{k-1}} + C$$

$$I_k = \int \frac{dt}{(t^2+a^2)^k} = \left| \begin{array}{ll} u = \frac{1}{(t^2+a^2)^k} & du = \frac{-2kt\,dt}{(t^2+a^2)^{k+1}} \\ dv = dt & v = t \end{array} \right| =$$

$$= \frac{t}{(t^2+a^2)^k} + 2k\int \frac{t^2\,dt}{(t^2+a^2)^{k+1}} = \frac{t}{(t^2+a^2)^k} + 2k\int \frac{(t^2+a^2)-a^2}{(t^2+a^2)^{k+1}}\,dt =$$

$$= \frac{t}{(t^2+a^2)^k} + 2k\int \frac{dt}{(t^2+a^2)^k} - 2ka^2\int \frac{dt}{(t^2+a^2)^{k+1}} = \frac{t}{(t^2+a^2)^k} + 2kI_k - 2ka^2 I_{k+1}$$

tj. $\quad I_k = \frac{t}{(t^2+a^2)^k} + 2kI_k - 2ka^2 I_{k+1}$

Odavde dobijamo rekurentnu formulu:

$$I_{k+1} = \frac{1}{2ka^2}\,\frac{t}{(t^2+a^2)^k} + \frac{2k-1}{2ka^2}\, I_k \,, \quad k = 1, 2, \dots$$

Na taj način ~~$I_1 =$~~ pošto je

$$I_1 = \int \frac{dt}{t^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{t}{a} + c$$

po rekurentnoj formuli dobijamo $I_2 = \int \frac{dt}{(t^2+a^2)^2}$ itd.

<u>Primjer</u> $\displaystyle\int \frac{x^4-3x}{x^3+1}\,dx$

$$R(x) = \frac{x^4-3x}{x^3+1} = \frac{x^4+x-4x}{x^3+1} = x + \frac{-4x}{x^3+1}$$

$$\begin{array}{r|l} x^4-3x & x^3+1 \\ -\;x^4+x & x \\ \hline -4x & \end{array} \qquad (x^4-3x) = (x^3+1)\cdot x - 4x$$

$$\frac{-4x}{x^3+1} = \frac{-4x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad \Big/ (x+1)(x^2-x+1)$$

$$-4x = A(x^2-x+1) + \cancel{B(x+1)}\,(Bx+C)(x+1)$$

$$\cancel{-4x = Ax^2 + (B-A)x + (A+B)}$$

$$\cancel{A = 0} \qquad \cancel{B - A = -4} \qquad \cancel{A + B =}$$

$$-4x = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$-4x = (A+B)x^2 + (-A+B+C)x + A + C$$

$$\begin{aligned} A + B &= 0 \\ -A + B + C &= -4 \\ A + C &= 0 \end{aligned} \qquad \begin{aligned} B &= -A \\ C &= -A \end{aligned} \;\Rightarrow\; \begin{aligned} -A - A - A &= -4 \\ A &= +4/3 \\ B &= -\tfrac{4}{3} \\ C &= -4/3 \end{aligned}$$

$$\int \frac{x^4-3x}{x^3+1}\,dx = \int x\,dx + \frac{4}{3}\int\frac{dx}{x+1} - \frac{4}{3}\int\frac{(x+1)\,dx}{x^2-x+1} =$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{4}{6}\ln|x^2-x+1| - \frac{4}{3}\,\frac{3}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C$$

Teorema Svaka prava racionalna funkcija se može zapisati kao konačan zbir prostih racionalnih funkcija tj.

$$R(x)=\frac{P_m(x)}{Q_n(x)}=\frac{A_{11}}{x-x_1}+\frac{A_{12}}{(x-x_1)^2}+\cdots+\frac{A_{1k_1}}{(x-x_1)^{k_1}}+\cdots+\frac{A_{s1}}{x-x_s}+\frac{A_{s2}}{(x-x_s)^2}+\cdots$$

$$+\cdots+\frac{A_{sk_s}}{(x-x_s)^{k_s}}+\frac{B_{11}x+C_{11}}{x^2+p_1x+q_1}+\frac{B_{12}x+C_{12}}{(x^2+p_1x+q_1)^2}+\cdots+\frac{B_{1m_1}x+C_{1m_1}}{(x^2+p_1x+q_1)^{m_1}}+$$

$$+\cdots+\frac{B_{l1}x+C_{l1}}{x^2+p_{l}x+q_l}+\cdots+\frac{B_{lm_l}x+C_{lm_l}}{(x^2+p_lx+q_l)^{m_l}}.$$

gdje je $k_1+k_2+\cdots+k_s+2(m_1+\cdots+m_l)=n$.

Integracija nekih iracionalnih funkcija

Integracija racionalnih funkcija

Funkcije oblika $R(u_1,u_2,\ldots,u_n)=\dfrac{P(u_1,u_2,\ldots,u_n)}{Q(u_1,u_2,\ldots,u_n)}$

gdje su P i Q polinomi od promjenljivih $u_1,u_2,\ldots,u_n$ se nazivaju racionalnim funkcijama od $u_1,u_2,\ldots,u_n$.

Ako su $u_1=\varphi_1(x),\ldots,u_n=\varphi_n(x)$, gdje su $\varphi_i(x)$ – neke funkcije, $i=1,2,\ldots,n$ to se funkcija $R(\varphi_1(x),\ldots,\varphi_n(x))$ naziva racionalnom funkcijom od $\varphi_i(x)$, $i=1,2,\ldots,n$.

Na primjer funkcija

$$f(x)=\frac{x+\sqrt[3]{(x^2+2)^2}+\sqrt[6]{(x-1)^5}}{x^2\left(1-\sqrt[5]{(x+1)^3}\right)}$$

je racionalna funkcija od x, $\sqrt[3]{x^2+2}$, $\sqrt[6]{x-1}$, $\sqrt[5]{x+1}$ tj.

$$f(x)=R\left(x,\sqrt[3]{x+2},\sqrt[6]{x-1},\sqrt[5]{x+1}\right)$$

Proučitimo funkcije iracionalnih fja su samo u rijedkim slučaje- vima elementarne funkcije. Osnovni metod računanja neodređ. integrala ~~ra~~ iracionalnih funkcija je metod racionalizacije podintegralne funkcije tj. traženje takve smjene koja ~~dovodi~~ prevodi dani integral u integral od racionalne funkcije. Govorićemo da smjena racionalizuje integral razmatrane funkcije.

Ovdje ćemo dati najvažnije slučajeve kada ~~smjena~~ ~~dovodi~~ uz pogodnu smjenu integral iracionalne f-je se svodi na integral racionalne f-je.

1) Integral ~~f-je~~ oblika $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_1}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{p_n}\right]dx$

Razmatraćemo integral ovog oblika, gdje su $p_1, p_2, .., p_n$ - racionalni brojevi. Neka je m najveći zajednički sadržalac ovih racionalnih brojeva tj. $p_i = \frac{q_i}{m}$, $i = 1, 2, .., n$, gdje su q_i cijeli brojevi

Uvodi se

SMJENA $\frac{ax+b}{cx+d} = t^m \Rightarrow \cancel{x}\cdot(cx+d)t^m = ax+b \Rightarrow$
$x(ct^m - a) = b - dt^m \Rightarrow$

$$\Rightarrow \quad x = \frac{t^m d - b}{a - ct^m}, \quad dx = m\frac{(ad-bc)t^{m-1}}{(a-ct^m)^2}dt$$

$$\left(\frac{ax+b}{cx+d}\right)^{p_1} = t^{q_1}, \quad \ldots, \quad \left(\frac{ax+b}{cx+d}\right)^{p_n} = t^{q_n}$$

Tada se naš integral pretvara u integral

$$\int R^*(t, t^{q_1}, .., t^{q_n})dt = \int R^*(t)dt$$

gdje je $R^*(t)$ - racionalna funkcija od t.

Primjer Izračunati $\int \frac{dx}{\sqrt{(x-1)^3(x-2)}}$

Rješenje Preobrazimo naš integral:

$$\int \frac{dx}{\sqrt{(x-1)^3(x-2)}} = \int \sqrt{\frac{x-2}{x-1} \cdot \frac{1}{(x-1)^2(x-2)^2}}\,dx = \int \sqrt{\frac{x-2}{x-1}} \cdot \frac{dx}{(x-1)(x-2)} =$$

Uvodimo smjenu $t = \left(\frac{x-2}{x-1}\right)^{1/2}$ $\left\| t^2 = \frac{x-2}{x-1} \right\|$ $x = \frac{2-t^2}{1-t^2}$ $dx = \frac{2t\,dt}{(1-t^2)^2}$

$$x - 1 = \frac{1}{1-t^2}, \quad x - 2 = \frac{t^2}{1-t^2}$$

$$\int \sqrt{\frac{x-2}{x-1}} \frac{dx}{(x-1)(x-2)} = \int \frac{2t^2(1-t^2)^2}{t^2(1-t^2)^2}dt = \int 2\,dt = 2t + C =$$

$$= 2\sqrt{\frac{x-2}{x-1}} + C$$

Primjer 2 $\displaystyle\int \frac{dx}{x\left(2+\sqrt[3]{\frac{x-1}{x}}\right)}$

Rješenje Smjena $t^3=\frac{x-1}{x}$, $x=\frac{1}{1-t^3}$ $dx=\frac{3t^2dt}{(1-t^3)^2}$

$$\int \frac{dx}{x\left(2+\sqrt[3]{\frac{x-1}{x}}\right)} = 3\int \frac{t^2dt}{(1-t)(2+t)(t^2+t+1)}$$

$$\frac{t^2}{(1-t)(2+t)(t^2+t+1)} = \frac{A}{1-t}+\frac{B}{2+t}+\frac{Ct+D}{t^2+t+1}$$

$$A=\frac{1}{9},\ B=\frac{4}{9},\ C=-\frac{1}{3},\ D=-\frac{2}{3}$$

$$\Rightarrow \int \frac{t^2dt}{(1-t)(2+t)(t^2+t+1)} = \int\left[\frac{1}{9}\,\frac{1}{1-t}+\frac{4}{9}\,\frac{1}{2+t}-\frac{1}{3}\,\frac{t+2}{t^2+t+1}\right]dt=$$

$$=\frac{1}{9}\int\frac{dt}{1-t}+\frac{4}{9}\int\frac{dt}{2+t}-\frac{1}{3}\int\frac{\frac{1}{2}(2t+1)+\frac{3}{2}}{t^2+t+1}\,dt=$$

$$=-\frac{1}{9}\ln|1-t|+\frac{4}{9}\ln|2+t|-\frac{1}{6}\ln|t^2+t+1|-\frac{1}{2}\int\frac{dt}{t^2+t+1}.$$

Pošto je $\displaystyle\int\frac{dt}{t^2+t+1}=\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt3}{2}\right)^2}=\frac{2}{\sqrt3}\,\text{arctg}\,\frac{2}{\sqrt3}\left(t+\frac{1}{2}\right)+c$

To je tada

$$\int\frac{dx}{x\left(2+\sqrt{\frac{x-1}{x}}\right)}=-\frac{1}{3}\ln\left|1-\sqrt[3]{\frac{x-1}{x}}\right|+\frac{4}{3}\ln\left|2+\sqrt[3]{\frac{x-1}{x}}\right|-$$
$$-\frac{1}{2}\ln\left(\sqrt[3]{\left(\frac{x-1}{x}\right)^2}+\sqrt[3]{\frac{x-1}{x}}+1\right)-\sqrt3\,\text{arctg}\,\frac{2}{\sqrt3}\left(\sqrt[3]{\frac{x-1}{x}}+\frac{1}{2}\right)+C$$

2) Integrali oblika $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$.

Racionalizacija ovih integrala se dobija pomoću jedne od Ojlerovih smjena:

1) $a>0$, tada $\sqrt{ax^2+bx+c}=\pm x\sqrt a\pm t\ /^2$

Ako kvadriramo $\Rightarrow$ $\cancel{ax^2}+bx+c=\cancel{ax^2}+2\sqrt a\,x t+t^2$

odavde je $x=\dfrac{t^2-c}{b-2\sqrt a\,t}$; $dx=\left(\dfrac{t^2-c}{b-2\sqrt a\,t}\right)'dt$

2) $c>0$, $\sqrt{ax^2+bx+c} = xt+\sqrt{c} \;/^2$

$$ax^2+bx+\cancel{c} = x^2t^2+2\sqrt{c}\,xt+\cancel{c} \;/\cdot\frac{1}{x}$$

$$ax+b+\cancel{\frac{c}{x}} = xt^2+2\sqrt{c}\,t$$

$$x = \frac{2\sqrt{c}\,t-b}{a-t^2}, \quad dx = \left(\frac{2\sqrt{c}\,t-b}{a-t^2}\right)'dt$$

3) Ako su x_1 i x_2 realne nule trinoma ax^2+bx+c tj. ako je $ax^2+bx+c = a(x-x_1)(x-x_2)$ to u tom slučaju je smjena

$$\sqrt{ax^2+bx+c} = (x-x_0)t$$

gdje je x_0 - jedna od nula ovog trinoma.

Primjer $\int \frac{1-\sqrt{x^2+x+1}}{x\sqrt{x^2+x+1}}\,dx$

Rješenje smjena $\sqrt{x^2+x+1} = xt+1$, $(c=1>0)$ smjena 2

Tada je $x = \frac{2t-1}{1-t^2}$ $dx = 2\,\frac{t^2-t+1}{(1-t^2)^2}\,dt$

$$\sqrt{x^2+x+1} = \frac{2t-1}{1-t^2}\cdot t+1 = \frac{2t^2-t+1-t^2}{1-t^2} = \frac{t^2-t+1}{1-t^2}$$

Integral se svodi na

$$I = \int \frac{1-\frac{t^2-t+1}{1-t^2}}{\frac{2t-1}{1-t^2}\cdot\frac{t^2-t+1}{1-t^2}}\,2\cdot\frac{t^2-t+1}{(1-t)^2}\,dt =$$

$$= 2\int \frac{\frac{1-t^2-t^2+t-1}{\cancel{1-t^2}}}{\frac{(2t-1)\cancel{(t^2-t+1)}}{\cancel{(1-t^2)^2}}}\cdot\frac{\cancel{t^2-t+1}}{\cancel{(1-t^2)^2}}\,dt = 2\int \cancel{t\,dt} = 2\int\frac{(t-2t^2)\,dt}{(2t-1)(1-t^2)} =$$

$$= 2\int\frac{t(1-2t)\,dt}{(2t-1)(1-t^2)} =$$

$$= +2\int\frac{t\,dt}{(t-1)(t+1)}$$

$$= \int \frac{2t\,dt}{(2t-1)(1-t)(1+t)}$$ i dalje rac. f-ja.

Primjer 2. $\int \frac{dx}{\sqrt{2x^2-6x+4}}$

Rješ. $2x^2-6x+4 = 2(x-2)(x-1)$

Smjena $\sqrt{2x^2-6x+4} = t(x-2)$ $\quad t = \frac{\sqrt{2x^2-6x+4}}{x-2}$

$$x = \frac{2t^2-2}{t^2-2} \quad dx = -\frac{4t\,dt}{(t^2-2)^2}, \quad \sqrt{2x^2-6x+4} = \frac{2t}{t^2-2}$$

$$\int \frac{dx}{\sqrt{2x^2-6x+4}} = -\int \frac{4t\,(t^2-2)\,dt}{2t\,(t^2-2)^2} = -2\int \frac{dt}{t^2-2} =$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{t+\sqrt{2}}{t-\sqrt{2}}\right| + C = \frac{1}{2}\,\frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{x-1}+\sqrt{x-2}}{\sqrt{x-1}-\sqrt{x-2}}\right| + C$$

Integracija diferencijalnog binoma

Izraz $x^m(a+bx^n)^p\,dx$ - naziva se diferencijalnim binomom, gdje su a i b - konstante, m, n i p racionalni brojevi. Ovdje imamo zadatak da izračunamo integral:

$$\int x^m(a+bx^n)^p\,dx$$

Razlikujemo tri slučaja:

1) $p \in \mathbb{Z}$, tada ~~je smjena~~ koristimo binomni obrazac.

2) $\frac{m+1}{n} \in \mathbb{Z}$, $p = \frac{r}{s}$ tada koristimo smjenu: $\cancel{t^s = ax^n+b}$ $t^s = a + bx^n$

3) $\frac{m+1}{n} + p \in \mathbb{Z}$ tada je smjena $t^s = \frac{a}{x^n} + b$
$p = \frac{r}{s}$,

Primjer
$$\int \frac{dx}{x^6\sqrt{x^2-1}} = \int x^{-6}\cdot(x^2-1)^{-1/2}\,dx = \left|\begin{array}{l} m=-6 \\ n=2 \\ p=-\frac{1}{2} \\ \frac{m+1}{n} = \frac{-6+1}{2} = \frac{-5}{2} \notin \mathbb{Z} \\ \frac{m+1}{n}+p = \frac{-6+1}{2} - \frac{1}{2} = \\ = -\frac{5}{2} - \frac{1}{2} = -\frac{6}{2} = -3 \in \mathbb{Z} \end{array}\right|$$

$$= \int x^{-6}\cdot x^{-1}\left(1-\frac{1}{x^2}\right)^{-1/2} dx = \left\|\begin{array}{l} t^2 = 1-\frac{1}{x^2} \\ 2t\,dt = \frac{2}{x^3}\,dx \\ \frac{1}{x^2} = 1-t^2 \\ x^2 = \frac{1}{1-t^2} \end{array}\right\| = \quad t = \frac{\sqrt{x^2-1}}{x}$$

$$= \int (x^2)^{-2}\cdot\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}}\cdot\frac{dx}{x^3} =$$

$$= \int (1-t^2)^2\cdot\frac{1}{t}\cdot t\,dt = \int (1-2t^2+t^4)\,dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + C$$

$$= \frac{\sqrt{x^2-1}}{x} - \frac{2}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{x^2-1}}{x}\right)^5 + C$$

Specijalni slučajevi. (ovaj dio do prije binoma)

Naš integral $\int R(x, \sqrt{ax^2+bx+c})\,dx$ možemo računati i drugim smjenama. Pošto je

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

to uzevši smjenu $x+\frac{b}{2a} = t$, to se naš integral svodi na jedan od tri sljedeća integrala:

$$\int R^*(t, \sqrt{1-t^2})\,dt\,; \quad \int R^*(t; \sqrt{t^2-1})\,dt, \quad \int R^*(t, \sqrt{t^2+1})\,dt$$

gdje je R^* racionalna f-ja. Ova tri integrala se rješavaju pomoću smjena

$$t = \sin u, \quad t = \frac{1}{\sin u}\,; \quad t = \operatorname{tg} u$$

ili hiperboličkih smjena

$$t = \operatorname{sh} u, \quad t = \operatorname{ch} u, \quad t = \operatorname{th} u$$

Integral oblika $\int R(\sin x, \cos x)\,dx$

R - racionalna f-ja promjenljivih $u_1 = \sin x$, i $u_2 = \cos x$.

Ovaj integral se racionalizuje univerzalnom trigonometrijskom smjenom $\boxed{\operatorname{tg}\frac{x}{2} = t}$.

Zaista, $x = 2\operatorname{arctg} t$, $dx = \dfrac{2dt}{1+t^2}$

$$\sin x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} \quad \left(\sin x = \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} \right) = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int R(\sin x, \cos x)\,dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}$$

Primjer Izračunati

$$\int \frac{dx}{5-4\sin x+3\cos x} = \left\| t = \operatorname{tg}\frac{x}{2} \right\| = \int \frac{2\,dt}{(1+t^2)\left(5-4\frac{2t}{1+t^2}+3\frac{1-t^2}{1+t^2}\right)} =$$

$$= \int \frac{2\,dt}{5(1+t^2)-8t+3(1-t^2)} = \int \frac{2\,dt}{2t^2-8t+8} = \int \frac{dt}{(t-2)^2} = \frac{1}{2-t}+C =$$

$$= \frac{1}{2-\operatorname{tg}\frac{x}{2}}+C$$

Ako je

1) $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ smjena $t = \cos x$

2) Ako je $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ smjena $t = \sin x$

3) Ako je $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ smjena $t = \operatorname{tg} x$

Integrali oblika $\int \sin\alpha x \cos\beta x\, dx$

Koriste se formule:

$$\cos\alpha x \cdot \cos\beta x = \frac{1}{2}\left(\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right)$$

$$\sin\alpha x \sin\beta x = \frac{1}{2}\left(\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right)$$

$$\sin\alpha x \cos\beta x = \frac{1}{2}\left(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x\right)$$

Na primjer $\int \sin 3x \sin 5x\, dx = \frac{1}{2}\int(\cos 2x - \cos 8x)\,dx = \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$

Integrali oblika $\int \sin^m x \cos^n x\, dx$

m - neparno $\quad t = \cos x$

n - neparno $\quad t = \sin x$

m,n - parno $\quad t = \operatorname{tg} x$

Ovdje se često koriste formule

$$\sin^2 x = (1-\cos 2x)/2$$

$$\cos^2 x = (1+\cos 2x)/2$$

Primjer $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x\,(1-\sin^2 x)\cos x\, dx =$

$$= \int(\sin^2 x - \sin^4 x)\cos x\, dx = \left\| \begin{matrix} t = \sin x \\ dt = \cos x\, dx \end{matrix} \right\| = \int(t^2-t^4)\,dt = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$$I_n = \int \sin^n x \, dx$$

$$I_n = \int \cos^n x \, dx$$

$$I_n' = \int \frac{dx}{\sin^n x}$$

$$I_n' = \int \frac{dx}{\cos^n x}$$

$$I_n = \int \sin^n x \, dx = \int \sin^{n-2} x \sin^2 x \, dx =$$

$$= \int \sin^{n-2} x \,(1-\cos^2 x)\, dx =$$

$$= \int \sin^{n-2} x \, dx - \int \sin^{n-2} x \cos^2 x \, dx = \quad \left| \begin{array}{ll} u = \cos x & du = -\sin x \, dx \\ dv = \sin^{n-2} x \cos x \, dx & \\ v = \frac{\sin^{n-1} x}{n-1} & \end{array} \right.$$

$$= I_{n-2} - \left[\frac{1}{n-1} \cos x \sin^{n-1} x + \frac{1}{n-1} \int \sin^{n-1} x \cdot \sin x \, dx \right] =$$

$$= I_{n-2} - \frac{1}{n-1} \cos x \sin^{n-1} x - \frac{1}{n-1} I_n \quad / (n-1)$$

$$n I_n = I_{n-2}(n-1) - \cos x \sin^{n-1} x$$

$$I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$$

$$I_0 = \int dx = x + c$$

$$I_1 = \int \sin x \, dx = -\cos x + c$$

$$I_5 = \int \sin^5 x \, dx =$$

$$= -\frac{1}{5} \cos x \sin^4 x + \frac{4}{5} I_3 =$$

$$= -\frac{1}{5} \cos x \sin^4 x - \frac{4}{15} \cos x \sin^2 x - \frac{8}{15} \cos x + c$$

Integral oblika $\int R(\operatorname{sh} x, \operatorname{ch} x)\, dx$

smjena $t = \operatorname{th} \frac{x}{2}$ $\quad \operatorname{sh} x = \frac{2t}{1-t^2}$, $\operatorname{ch} x = \frac{1+t^2}{1-t^2}$

$$dx = \frac{2t}{1-t^2}$$

1) Izračunaj: $\displaystyle\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\,dx =$

$p_1 = \frac{2}{3}$, $p_2 = \frac{1}{6}$, $p_3 = \frac{1}{3}$

$m = 6$ smjena $x = t^6$

$dx = 6t^5dt$ $\quad t = \sqrt[6]{x}$

$$= 6\int \frac{t^6+t^4+t}{t^6(1+t^2)}\,t^5dt = 6\int \frac{t^5+t^3+1}{1+t^2}\,dt =$$

$$= 6\int\left(t^3 + \frac{1}{t^2+1}\right)dt = \frac{3}{2}t^4 + 6\,\mathrm{arctg}\,t + c = \frac{3}{2}\sqrt[3]{x^2} + 6\,\mathrm{arctg}\,\sqrt[6]{x} + c$$

2) Izračunaj $\displaystyle\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \int x^{-1/2}\left(1+x^{1/4}\right)^{1/3}dx$

$m = -\frac{1}{2}$, $n = \frac{1}{4}$, $p = \frac{1}{3}$

$\frac{m+1}{n} = 2$ smjena je $\boxed{t^3 = \left(1+x^{1/4}\right)}$

$$x = (t^3-1)^4 \quad dx = 12t^2(t^3-1)^3dt$$

$$\int = 12\int (t^3-1)^{-2}\,t\cdot t^2(t^3-1)^3dt = 12\int(t^6-t^3)dt =$$

$$= \frac{3}{7}t^4(4t^3-7)+c = \frac{3}{7}\left(1+\sqrt[4]{x}\right)^{1/3}\left[4\left(1+\sqrt[4]{x}\right)-7\right]+c$$